Discipline Course-I

Semester-II

Paper No: Electricity and Magnetism

Lesson: Electromagnetic Induction Lesson 5.2: Self and Mutual Inductance

Lesson Developer: Mr. N K Sehgal
College/ Department: Hans Raj College, University of Delhi

Table of Contents

Chapter: Self and Mutual Inductance

Learning Objectives

After going through this chapter, the reader will,

i. Be able to understand the meaning and significance of the phenomenon of self inductance
ii. Know the meaning of the term, ‘co-efficient of self inductance’
iii. Understand the meaning and know the definition of henry, the SI unit of the coefficient of self inductance
iv. Know the general method of calculating the coefficient of self inductance (L)
v. Be able to calculate, L, for a solenoidal coil
vi. Appreciate why the coefficient of self inductance can be regarded as a measure of the ‘electrical inertia’ of a given system
vii. Know the expression for the energy needed to establish a current (say I) in a given system
viii. Know the formulae for calculating the equivalent ‘inductance’ of a number of inductors connected (i) in series and (ii) in parallel
ix. Know the factors that affect the coefficient of self inductance
x. Be able to understand the meaning and significance of the phenomenon of ‘mutual induction’
xi. Know the meaning of the term ‘coefficient of mutual inductance’ for a given pair of coils
xii. Know the factors that affect the ‘coefficient of mutual inductance’.
xiii. Understand the meaning and know the definition of henry , the SI unit of the coefficient of mutual inductance
xiv. Know the general method of calculating the coefficient of mutual inductance (M)
xv. Be able to calculate M for a pair of solenoidal coils.
xvi. Know the meaning and the relevant expression for the ‘coefficient of coupling’ for a given pair of coils
xvii. Be able to do problems based on the phenomena of self and mutual inductance
xviii. Know the significance of the ‘reciprocity theorem’ for the coefficient of mutual inductance for a given pair of coils
xix. Understand why the magnetic field can be regarded as a ‘seat of energy’
xx. Obtain an expression for the energy density of the magnetic field.

Introduction

We start this chapter by explaining the meaning and significance of the phenomenon of self induction. The reader is introduced to the meaning and significance of the ‘coefficient of self inductance (L) for a given coil or circuit. The Henry, the unit of L is introduced, defined and correlated with the other known units. The factors affecting the coefficient of self inductance are discussed in some detail. The properties associated with L are also discussed. An expression for the energy needed to establish a given current I, in a coil or circuit, of self inductance L, is derived. The formulae for the equivalent inductance of a number of inductors connected in (i) series and (ii) parallel are stated and compared with the corresponding formulae for resistors.

The student is next introduced to the phenomenon of mutual inductance for a given pair of coils/circuits. The meaning of the term coefficient of mutual inductance for a given pair of coils is discussed and explained in some detail.

A very important theorem, the reciprocity theorem, for a given pair of coils, is discussed and explicitly proved. The significance of this theorem is clearly brought out.

We close this chapter by discussing how the magnetic field can be regarded as a 'seat of energy'. The expression for the 'energy density' of the magnetic field is stated. Its similarity with the corresponding expression for the energy density of the electric field is clearly brought out.

Self Inductance

We know that

- A current carrying coil produces a magnetic field
- The magnetic field lines form appropriate closed curves
- Whenever the total number of magnetic field lines linked normally with a coil/circuit changes (i.e. the magnetic flux linked with the coil/circuit changes), an induced e.m.f is produced in it as per the phenomenon of electromagnetic induction.

It follows that any coil/circuit would have a self induced e.m.f setup in it whenever the current flowing in it changes. This change can occur, even for a steady d-c voltage source, at the instants the circuit is just closed or opened. We, therefore, would have a self induced e.m.f in a coil/circuit, at least for some (may be quite small) time interval.

The self induced e.m.f, as per Lenz's law, would try to oppose the very cause to which it is due. The self induced e.m.f can therefore, be regarded as an attempt by the coil/circuit to maintain 'status quo' viz-a-viz its electrical state.

We call this phenomenon of having a self induced e.m.f and thereby opposing any change in the electrical state of a coil or circuit, as the phenomenon of self inductance.

Coefficient of Self Inductance

From Biot-Savart law, namely,

$$d\mathbf{B} = \mu_0/4\pi \; I \; d\mathbf{l} \times \mathbf{r}/r^3$$

we know that the magnitude of the magnetic field ($d\mathbf{B}$), due to a current carrying element ($d\mathbf{l}$), would be always proportional to the current flowing through the coil.

The magnetic flux, linked with any circuit/coil (given by $\phi = \int_s \mathbf{B}.d\mathbf{s}$) is clearly proportional to the strength of the magnetic field there. The latter, in turn, is proportional to the current, I, that caused it. We can, therefore, say:

The magnetic flux (ϕ), linked with any coil/circuit, at any instant, would be proportional to the current (I) flowing through it at that instant. Mathematically, it implies that

$$\phi \propto I$$

We can, therefore, put, $$\phi = L\,I$$

The constant of proportionality, L, would clearly be a characteristic of the coil/circuit involved. We call 'L' as the coefficient of self inductance for the given coil/circuit.

The above relation gives us a definition of L. we have, $\phi = L$ when $I = 1$ unit.

Thus the coefficient of self inductance, of a given coil/circuit, may be defined as the magnetic flux linked with it due to a unit current flowing through the coil/circuit itself.

From this definition, we find that the SI unit of 'L' would be Wb/A.

Again, from $\phi = L I$, we get,

$$\varepsilon = - d\phi/dt = - L\, dI/dt$$

It follows that $|\varepsilon| = L$ if $dI/dt = I$ unit. We may therefore, also define, L, as follows:

The coefficient of self inductance of a given coil/circuit equal the magnitude of the emf induced in it when the rate of change of current through the coil/circuit is unity.

From this definition, we find that,
the SI unit of 'L' may also be expressed as volt/(ampere/s) = s- volt/ampere = ohm-s.

The SI Unit for the Coefficient of Self Inductance

The SI unit for the coefficient of self inductance has been named as henry (H). This name has been given in honor of Joseph Henry, the American scientist who did a lot of work (independent of Faraday) on the phenomenon of electromagnetic induction.

We can now say:

A coil/circuit would be said to have a self inductance of one henry, if the magnetic flux linked with it, due to a current of one ampere in itself, equals one weber (Wb).

Alternatively, we can also say:

A coil/circuit, would be said to have a self inductance of one henry is the induced emf, set up in it, when the current, in it itself, is changing at the rate of 1 As^{-1}, equal one volt.

It follows that, henry = weber/ampere of $H = Wb\, A^{-1}$ and

Henry = volt/ (ampere/s) or H = ohm-s = (volt/ampere)-s

General Method for Calculating L

A general method for calculating L, can be based on the defining relation:

$$\phi = L I$$

It gives, $L = \phi/I$

We can therefore proceed as follows:

(i) We imagine a current I to flow through the coil/circuit.

(ii) We next calculate the magnetic field (B), due to this current, at any area element, ds, of the coil/circuit.

(iii) The magnetic flux, linked with the coil/circuit, is, therefore, $\phi = \int_s B.ds$

(iv)We finally ut $\phi = L\ I$. The term I gets cancelled on both sides and the coefficient of self inductance gets expressed in terms of the characteristics of the given coil/circuit.

In practice, this general method can however, be used in but a few cases. This is because of the difficulty of calculating B (Using either the Biot-Savart law or Ampere's circuital law) for an arbitrary, general, current distribution.

Coefficient of Self Inductance for a (long thin) Solenoidal Coil

Consider a long thin solenoid wound with n turns of closely packed insulated wire, per unit of its length. If this solenoid were to carry a current I, the axial magnetic field, at points far away from its end, would be given by

$$B = \mu_0 nI$$

The field would vary at points other than its axial points. However, for a log thin solenoid, we can, as a first approximation, neglect this variation and consider this magnetic field value to be valid at all points located on a cross section of the solenoid.

The magnetic flux, linked with one turn (cross-sectional area=πa^2 where a=radius of the solenoid), would be:

$$\phi_1 = (\pi a^2) \times B = \pi a^2 \times \mu_0 nI = \mu_0 n\ \pi a^2 I$$

The magnetic flux, linked with a unit length would, therefore, be:

$$\phi = n\ \phi_1 = \mu_0 n^2\ \pi a^2 I$$

Therefore, $L = \phi/I = \mu_0 n^2 \pi a^2$

<u>Example 1</u>:

A magnetic field **B**, directed along the z-axis, varies with,

(i) position of the field point along the x-axis and
(ii) time,t. The rate of change of field, with x, is $\left(\frac{dB}{dx}\right)_t$= a, say.

The rate of change of field, with time is $\left(\frac{dB}{dt}\right)_x$ = b.
A square loop, of side l, with its two sides parallel to the x and y axis, lies in the x-y plane. If it were to move with a speed v, parallel to the x-axis, find the induced e.m.f in the loop.

<u>Solution</u>:

The induced e.m.f here is due to two causes:

(i) due to variation of B, with t, at a given point in space

(ii) due to the motion of the loop in a (normal to its plane) magnetic field.

The rate of change of flux, due to cause (i), is

$$\frac{d\phi_1}{dt} = (l^2)\frac{dB}{dt} = l^2 a$$

Where, l^2 is the area of the loop.

To find the rate of change of flux, due to the second cause, we write,

$$\frac{d\phi_2}{dt} = \frac{d}{dt}(l^2B) = (l^2)\frac{dB}{dt} = (l^2)\frac{dB}{dx}\cdot\frac{dx}{dt} = l^2(b).v \qquad \left(\text{since } \frac{dB}{dx} = b \text{ and } \frac{dx}{dt} = v\right)$$

Therefore, total induced e.m.f = $-\left(\frac{d\phi_1}{dt} + \frac{d\phi_2}{dt}\right) = -l^2(a+bv)$

Self Inductance as 'Electrical Inertia'

From the relation, $\varepsilon = -L\frac{dI}{dt}$,

it is easy to appreciate that more the value of L, for a given coil/circuit, the more is the self induced e.m.f in it (for a given rate of change of current through it). And the more this self induced e.m.f, the more difficult it would be to change the electrical state of the given coil/circuit. This is so because, as per Lenz's law, the induced e.m.f always opposes the change that caused it. Thus if the induced e.m.f is due to an increase in current through the coil/circuit, it would tend to decrease this current and vice-versa. Hence as per Lenz's law, the self induced e.m.f always tries to maintain the current in the circuit at the value it had, i.e., it tries to maintain the status-quo viz-a-viz the electrical state of the system.

We recall that a similar situation exists in mechanics. The more the mass of a given object/system, the more difficult it is (for an external force) to change the state of motion of the object/system.

We can now appreciate that, in mechanics, mass is a measure of the mechanical inertia of a given object/system. In a similar way, the coefficient of self inductance can be viewed as a measure of the electrical 'inertia' of a given system (coil/circuit) i.e., a measure of its ability to oppose any change of current through it.

Inertia, by definition, implies maintaining status quo or 'opposing any change'. The above discussion, therefore, justifies regarding 'L' as a measure of the electrical inertia of a given coil/circuit.

Energy needed to establish a current (I) in a given system

Let a coil/circuit, having a coefficient of self inductance L, have a current, I, flowing through it any instant, t. If this current were to change by an amount dI, in a time dt, the instantaneous induced e.m.f would be:

$$\varepsilon = -L\frac{dI}{dt}$$

An amount of charge, dQ (= $\left(\frac{I+(I+dI)}{2}\right)$ dt = Idt nearly) flows though the coil/circuit in this small time interval. This self induced emf (ε) would tend to oppose this change in current. The main source (battery etc.) supplying current to the circuit/coil, would therefore have to do some work dW against this self induced e.m.f.

By definition, we then have:

$$dW = |\varepsilon|(Idt) = L\frac{dI}{dt}Idt = L\,I\,dI$$

The total work done (W), in changing the current from 0 to I, in the given coil/circuit, is, therefore given by:

$$W = \int_0^I LIdt = \frac{1}{2} LI^2$$

We, therefore, say that the main source needs to spend energy equal to ½ LI^2, in establishing a current I in a coil/circuit of self inductance L.

Where does this energy get stored? It is usual to say that this energy 'gets stored' in the magnetic field, say B, that gets produced, in the region around the coil/circuit as a result of the current, I, flowing in it.

The magnetic field can, therefore, be regarded as a store-house of energy. In a subsequent article, we would talk about the energy-density (energy/volume) of the magnetic field.

Combination of Inductances

We can combine two (or more) coils, each having its own characteristic value for the coefficient of self inductance. This combination can be done in two basic simple ways:

(i) Combining inductances in series

(ii) Combining inductances in parallel.

It can be shown that the equivalent inductance of the combination in the two cases is given by the following formulae:

(i) $L = \sum_{i=1}^{n} L_i$

for inductances joined in series.

(ii) $1/L = \sum_{i=1}^{n} 1/L_i$

for inductances joined in parallel.

Here L_i (i=1 to n) are the values of the coefficient of self inductance for each of the individual inductances and L is the equivalent inductance of the combination.

We thus observe that the basic rules, for finding the equivalent inductance of a combination are the same as the corresponding rules for resistors. They are, however, not the same as the corresponding rules for capacitors.

Factors Affecting the Coefficient of Self Inductance

The coefficient of self inductance, of a given coil/circuit, is an indicator of the extent to which the magnetic field lines, due to a current in the coil/circuit itself, get linked with the coil/circuit. It would, therefore, depend on the geometry of the coil/circuit. It is easy to realize that a coiled arrangement is likely to entrap a greater number of magnetic field lines than a straight wire like set up.

Besides the geometry, the nature of the medium, associated with the coil/circuit, also affects its coefficient of self inductance. Thus the use of an insulated ferromagnetic material, as the core of a coil, is likely to cause a much greater number of magnetic field lines to get linked with it. This in turn would lead to an increase in the value of the coefficient of self inductance.

We can, therefore, say that the coefficient of self inductance depends on:

(i) Geometry of the coil/circuit

(ii) The nature of the medium associated with the coil/circuit.

Mutual Inductance

The phenomenon of mutual inductance is, in a way, the phenomenon that led to the discovery of electromagnetic induction. When there are two coils in the neighborhood of each other, a change of current, in one of them, would cause an induced e.m.f in the other. This induced e.m.f would, as already stated, last only as long as the current in the other coil is changing.

We can say:

The phenomenon of mutual induction implies the production of an induced e.m.f in one coil, due to changes of current in another neighboring coil. The induced e.m.f, as always, would try to oppose the very cause to which it is due.

Coefficient of Mutual Inductance

Let there be two coils, say 1 and 2, in the neighborhood of each other. The magnetic flux, ϕ_{21}, linked with coil 2, due to a current I_1 in coil 1, would, as already explained, be proportional to I_1. We can, therefore, write:

$$\phi_{21} \propto I_1$$

Or, $\phi_{21} = M_{21}I_1$

The constant of proportionality would depend on the two coils involved in the phenomenon. The constant M_{21} can be referred to as the coefficient of mutual inductance of coil 2 with respect to coil 1. It can be defined as follows:

The coefficient of mutual inductance of coil 2 with respect to coil 1, equal the magnetic flux linked with coil 2, due to a unit current in coil 1.

We also have, $\varepsilon_{21} = -\frac{d\phi_{21}}{dt} = -M_{21}\frac{dI_1}{dt}$

We can, therefore, say:

The coefficient of mutual inductance, of coil 2 with respect to coil 1, equals the magnitude of the e.m.f, induced in coil 2, when the current in coil 1 is changing at a unit rate.

Interchanging the roles of the two coils, we can write:

$$\phi_{12} = M_{12}I_2$$

and, $\varepsilon_{12} = -\frac{d\phi_{12}}{dt} = -M_{12}\frac{dI_2}{dt}$

The constant M_{12} can be referred to as the coefficient of mutual inductance of coil 1 with respect to coil 2. It can be defined in each of the two ways given above for defining M_{21}.

An interesting query that can come up at this stage is concerned with the equality or otherwise of the two coefficients M_{21} and M_{12}. We shall soon see that these two coefficients are always equal to each other, i.e.

$$M_{21} = M_{12} = M \text{ (say)}$$

The constant M is, therefore, a characteristic of the pair of coils involved. We can define M as follows:

The coefficient of mutual inductance, for a given pair of coils, equals,

(i) the magnetic flux linked with one of them due to the flow of a unit current in the other.

Or

(ii) the magnitude of the e.m.f induced in one of them when the current in the other is changing at a unit rate.

Factors Affecting the Coefficient of Mutual Inductance for a given pair of coils

The coefficient of mutual inductance for a given pair of coils, would (as in the case of coefficient of self inductance) depend on the,

(i) geometry of both the coils

(ii) nature of the medium between them

In addition to these two factors, this coefficient would also depend on the distance of separation between the given pair of coils. This is because the magnetic field, at the location of one of them, due to a current in the other, would decrease with an increase in the distance between them.

We can therefore, say that the coefficient of mutual inductance, for a given pair of coils, depends on:

(i) the geometry of both the coils

(ii) the nature of the medium associated with the given pair of coils

(iii) the distance of separation between the given pair of coils.

The SI Unit for the Coefficient of Mutual Inductance

The SI unit, for the coefficient of mutual inductance, would again be the henry (H). We can now say that:

The coefficient of mutual inductance, for a given pair of coils, equals one henry when the

(i) magnetic flux linked with one of them due to a current of 1 A in the other equal one weber.

Or

(ii) magnitude of the emf induced in one of them, due to a rate of change of current of 1 A/s, in the other, is one volt.

We may thus again say:

$$1H = 1\ Wb/A = 1\ V/As^{-1} = 1\ \Omega s$$

Joseph Henry

Joseph Henry, was an American scientist and a founding member of the National Institute for Promotion of Science. He was highly regarded during his lifetime. While building electromagnets, Henry discovered the phenomena of self inductance. He also discovered mutual inductance independently of Michael Faraday, although Faraday was the first to publish his results. Henry developed the electromagnet into a practical device. He invented a precursor to the electric doorbell and electric relay. Henry's work on the electromagnetic relay was the basis of the practical electrical telegraph.

General Method of Calculating M

The general method for calculating M, would again be as follows:

(1) We imagine a current I to flow through one of the coils.

(2) We next calculate the magnetic field (**B**), due to this current, at any area element d**s**, of the other coil.

(3) The magnetic flux, linked with the other coil is:

$$\phi = \int_s \mathbf{B}.d\mathbf{s}$$

(4) We finally put ϕ = MI. The term, I, gets cancelled on both sides and we get an expression, for M, in terms of the geometry/medium characteristics of the given pair of coils.

An important point, that needs to be noted here, is the following. We should prefer to choose that coil as the 'first coil' for which the magnetic field due to a current in it, can be more conveniently calculated. Also, the 'other coil' should preferably be that for which the magnetic field at all of its area elements has nearly the same value.

For example, while calculating the value of M, for a pair of vastly different sized concentric circular coils, we can take the smaller coil as the other coil.

Coefficient of Mutual Inductance for a Pair of Solenoidal Coils

Consider a pair of coils in which one insulated coil is wound over the other insulated coil on a long thin solenoidal core. The magnetic field lines, due to current in one coil, can then be thought to almost completely link with the other.

Let n_1 and n_2 be the number of turns per unit length for coil 1 and coil 2 respectively. Let r be the nearly common radius of either coil. If a current I_1 flows through coil 1, the magnetic field due to it, near its axis is:

$$B_1 = \mu_0 n_1 I_1$$

Because the solenoids are assumed to be long and thin, we can take the magnetic field to have a nearly constant value (the value of the axial field) at all points over the cross section of coil 2. Hence the magnetic flux linked with coil 2, is:

$$\phi_{21}= [(\pi r^2)\, B_1].(n_2 l)$$

Here $n_2 l$ represents the total number of turns of coil 2 over a length l of it.

Therefore,

$$\phi_{21}= (\pi r^2)\,(n_2 l)(\mu_0 n_1 I_1)$$

$$= (\mu_0 n_1 n_2)\,(\pi r^2 l)\, I_1$$

But, $\phi_{21}= M_{21} I_1$

Therefore,

$$M_{21} = (\mu_0 n_1 n_2)\,(\pi r^2 l)$$

This would also be the value of M_{12}. Hence the coefficient of mutual inductance per unit length of the two solenoidal coils is:

$$M_{21} = M_{12} = M \text{ (say)} = (\mu_0 n_1 n_2)\,(\pi r^2)$$

In the case where there is a medium of relative permeability μ_r, in the region where the two coils are present, we would have:

$$M = (\mu_r)\,(\mu_0 n_1 n_2)\,(\pi r^2)$$

$$= \mu n_1 n_2 \pi r^2$$

Where μ ($=\mu_r\mu_0$) stand for the permeability of the medium.

The above result holds for the mutual inductance per unit length.

For a length l, we would have:

$$M = \mu n_1 n_2\,(\pi r^2)(l)$$

It is thus seen that the mutual inductance depends on the geometry of the two coils and the nature of the medium separating them.

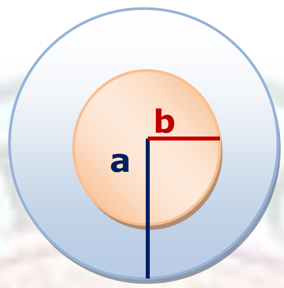
Example 2:

Two concentric coplanar coils have radii, a and b, respectively, with a>>b. Find an expression for their coefficient of mutual inductance.

Solution:

We assume a current I to flow through the larger coil. The magnetic field, at its centre is then given by:

$$B_0 = \mu_0 I/2a$$



This field is directed normal to the plane of both the coils. The smaller coil would have this field directed normal to its cross section. The radius of this coil being very small (in comparison to that of the larger coil), we can assume the field to have a value equal to B0 all over its cross section.

Hence, the magnetic flux, linked with the smaller coil, is:

$$\phi_{ba} = \pi b^2 (B_0) = \mu_0 I\ \pi b^2/2a$$

but,

$$\phi_{ba} = M_{ba} .I$$

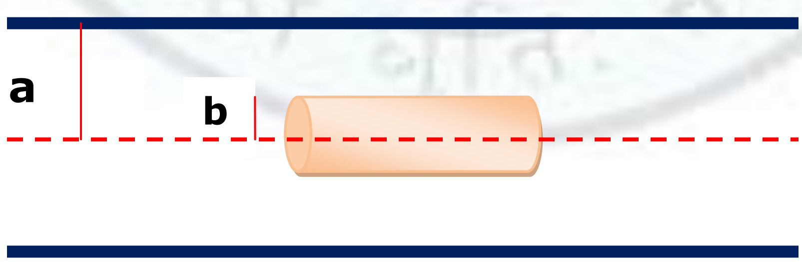
Since, $M_{ba} = M_{ab}$ (= M, say), we can say that the coefficient of mutual inductance, for the given pair of coils, is given by:

$$M = \mu_0 \pi b^2/2a$$

Example 3:

A small solenoid, of radius b, is place co-axially and near the centre of a large solenoid of radius a. Find the coefficient of mutual inductance for this pair of solenoidal coils.

Solution:



Imagine a current I to flow through the larger coil. The axial magnetic field, near the centre, is then

$$B = \mu_0 n_a I$$

where n_a is the number of turns per unit length of the larger solenoid.

The magnetic flux linked with the smaller solenoid, due to this magnetic field, is

$$\phi_{ba} = B\,\pi b^2 n_b l$$

where n_b is the number of turns per unit length of this solenoid and l is its total length.

Here we have again assumed the magnetic field to have a constant value all over the cross section of the smaller solenoid.

Since, $\phi_{ba} = M_{ba} I$

we have,

$$(\mu_0 n_a I)(\pi b^2)(n_b l) = M_{ba} I$$

Therefore,

$$M_{ba} = (\mu_0 n_a n_b)\pi b^2 l$$

As $M_{ab} = M_{ba} = M$ (say), we have

$$M = (\mu_0 n_a n_b)\,\pi b^2 l$$

It is important to note that in this (as well as in the previous) example, we selected the larger of the two coils as the current carrying coil. This enabled us to assume the magnetic field to have a nearly constant value all over the cross section of the smaller coil and calculate the magnetic flux linked with it. It would clearly not be easily possible to do this kind of a calculation for the larger coil, if we had assumed the smaller coil as the current carrying coil.

Coefficient of Coupling

Consider two coils placed in close proximity to each other. If a current flows in one of them, the field lines of the resulting magnetic field are then likely to be linked with the second coil to a very large extent. In an ideal case, we can think of this linkage as complete, i.e., all the magnetic field lines of the magnetic field due to the first coil, pass through the second coil. In such a case, the reverse would also be true.

It turns out that the extent of this magnetic coupling of two coils, depends on the values of (i) their coefficient of mutual inductance (M) and (ii) the coefficient of self inductance of each of the two coils (say L_1 and L_2). If $M^2 = L_1 L_2$, the magnetic coupling of the two coils is complete or perfect. In general, we do not have such an ideal situation. We, therefore, define a term K, given by:

$$K = M/\sqrt{L_1 L_2}$$

as the coefficient of coupling between a given pair of coils. Clearly K can approach 1 but would never be quite equal to 1. This implies that we would have

$$M \le \sqrt{L_1 L_2}$$

The Reciprocity Theorem

We have already noted that for a given pair of coils, $M_{12} = M_{21}$, i.e., the two values of mutual inductance are equal in pairs. This result is known as the reciprocity theorem. We may say that, as per the reciprocity theorem:

For a given pair of coils, say 1 and 2, (irrespective of their shapes or sizes), the magnetic flux linked with coil 2,due to a unit current in coil 1, is equal to the magnetic flux, linked with coil 1 due to a unit current in coil 2.

The reciprocity theorem can be proved in a simple way as follows. Let a current I_1, flowing through coil 1, produce a magnetic field $\mathbf{B}_1$, at a surface area element $d\mathbf{s_2}$ of coil 2. The total magnetic flux, linked with coil 2, would then be:

$$\phi_{21} = \int_{s2} \mathbf{B_1}.d\mathbf{s_2}$$

$$= \int_{s2} \text{curve } \mathbf{A_1}.d\mathbf{s_2}$$

where $\mathbf{A_1}$ is the vector potential associated with the field $\mathbf{B_1}$. By Stokes' theorem, we have

$$\int_{s2} \text{curve } \mathbf{A_1}.d\mathbf{s_2} = \int_{l2} \mathbf{A_1}.d\mathbf{l_2}$$

where l_2 is the curve enclosing the area s_2. Further, since

$$\mathbf{A} = \mu_0 I_1/4\pi \int_{l1} d\mathbf{l_1}/r$$

we get,

$$\int_{l2} \mathbf{A_1}.d\mathbf{l_2} = \int_{l2} \mu_0 I_1/4\pi \int_{l1} d\mathbf{l_1}/r \,.\, d\mathbf{l_2}$$

$$= \mu_0 I_1/4\pi \int_{l2} \int_{l} d\mathbf{l_1}.d\mathbf{l_2}/r$$

Here r is the magnitude of $\mathbf{r_b}$ ($\mathbf{r}$ is the position vector of the length element dl_2 with respect to the length element dl_1). Hence,

$$\phi_{21} = \mu_0 I_1/4\pi \int_{l2} \int_{l1} d\mathbf{l_1}.d\mathbf{l_2}/r$$

But,

$$\phi_{21} = M_{21} I_1$$

Hence,

$$M_{21} = \mu_0/4\pi \int_{l2} \int_{l1} d\mathbf{l_1}.d\mathbf{l_2}/r$$

In a similar way, we would also get,

$$M_{12} = \mu_0/4\pi \int_{l1} \int_{l2} d\mathbf{l_2}.d\mathbf{l_1}/r$$

This proves the reciprocity theorem. The formula for M_{21} or M_{12}, namely,

$$M_{21} = M_{12} = \mu_0/4\pi \int_{l2} \int_{l} d\mathbf{l_1}.d\mathbf{l_2}/r$$

is known as the **Newmann formula** for the coefficient of mutual inductance of a given pair of coils.

Energy Density of the Magnetic Field

We have already seen that the energy spent by a source in establishing a current I, in a coil of self inductance L, equals ½ LI^2. This energy can be thought of as getting stored in the magnetic field (**B**) produced by the current I, in the coil.

Now,

$$W = \frac{1}{2} L I^2 = \frac{1}{2} \phi I$$

where ϕ (=L I) is the magnetic flux linked with the coil, due to current I in itself.

By definition, $\phi = \int_s \mathbf{B}.d\mathbf{s}$

$= \int_s \text{curve } \mathbf{A}.\, d\mathbf{l}$

where, **A** is the vector potential associated with the magnetic field **B**.

Using Ampere's circuital law, (namely, curve $\mathbf{B} = \mu_0\mathbf{j}$) along with appropriate rules of vector algebra and relevant physical considerations, it can be shown that,

$$W = 1/2\mu_0 \int_\tau B^2 d\tau$$

This implies that we can think of a energy, $(1/2\mu_0)\, B^2$, as stored per unit volume of the region in which the magnetic field exists.

Thus energy density of magnetic field = $B^2/2\mu_0$

This result is similar to the corresponding result for electric field. The energy density of the electric field equals ½ $\varepsilon_0 E^2$.

Summary

- The phenomenon of self inductance implies the production of an induced e.m.f in a coil/circuit, due to a change in current in the circuit itself.
- The coefficient of self inductance, of a coil/circuit, equals the magnetic flux linked with it, due to a unit current in it.
- It also equals the e.m.f induced in the coil/circuit, due to a unit rate of change of current in the coil/circuit itself.
- The SI unit of coefficient of self inductance is the henry (H). A coil/circuit would have a self inductance of 1 H (one henry) if (i) the magnetic flux linked with it due to a current of 1 A in itself, equal one weber or if (ii) the magnitude of the e.m.f, induced in the coil is one volt when current in the circuit itself changes at the rate of $1As^{-1}$.
- 1 H = 1 (Wb/A) = 1 (V/As^{-1}) = 1 Ω-s.
- The coefficient of self inductance (L) of a solenoidal coil of radius a and length l, is given by $L = \mu_0 n^2 \pi a^2 l$ where n is the number of turns per unit length of the coil.
- We can regard self inductance as the electrical inertia associated with a given coil/circuit.
- The work done by the source in establishing a current I, in a coil of self inductance L, equals ½ L I^2. We regard this work as stored as the energy of the magnetic field produced by the current I, in the coil/circuit.

- The equivalent inductance of a number of coils, connected in
  (i) series, is given by : $L = \sum_{i=1}^{n} L_i$

  (ii) parallel, is given by: $1/L = \sum_{i=1}^{n} 1/L_i$

- The coefficient of self inductance of a coil/circuit depends on the (i) geometry of the coil/circuit and (ii) nature of the medium associated with the coil/circuit.

- The phenomenon of mutual inductance for a pair of coils implies the production of an induced e.m.f, in one of them, due to a change of a current in the other neighbouring coil.

- The coefficient of mutual inductance for a given pair of coils equals the:
  (i) flux linked with one of them due to a unit current in the other
  (ii) e.m.f induced in one of them, when the rate of change of current in the other is unity i.e., 1A/s.

- For a given pair of coils, the coefficient of mutual inductance are equal in pairs i.e., $M_{21}=M_{12}$.

- The coefficient of mutual inductance for a given air of coils equals one henry, if the:
  (i) magnetic flux linked with one of them, due to current of 1 A in the other, is one weber.
  (ii) Magnitude of e.m.f induced in one of them, due to a rate of change of current of 1 A s^{-1} in the other equal 1 Wb.

- For a pair of solenoidal coils, having a nearly common radius r, the coefficient of mutual inductance (M), for a length l, is given by:
  $M = (\mu_0 n_1 n_2)\pi r^2 l$

- The coefficient of coupling (K,) for a given pair of coils, is a measure of the extent to which the magnetic field lines, due to a current in one of them, get linked with the other coil. We define K as:

  $$K = M/\sqrt{L_1 L_2}$$

  Here L_1 and L_2 are the coefficients of self inductance for the two coils and M is their coefficient of mutual inductance.

- The equality,
  $M_{12} = M_{21}$ for a given pair of coils (irrespective of their shape, size or distance of separation) is known as the reciprocity theorem.

- We can write,
  $M_{21} = M_{12} = \mu_0/4\pi \int_{l2}\int_{l} d\mathbf{l_1}.d\mathbf{l_2}/r$
  this result is known as Newmann's formula.

- We regard the magnetic field as a seat of energy.

- The energy density, i.e., energy per unit volume, of the magnetic field, at a point where the magnetic field has a value B, is given by:

  Energy density of magnetic field = ½ B^2/μ_0.

Exercise

Fill in the Blanks

Fill in the blanks in the following statements with appropriate words/expressions:

(i) The phenomenon of self inductance implies the linkage of some magnetic flux, with the coil, due to a flow of ____________, in the coil itself.

(ii) We have 1 H = 1 Wb/_____ = 1 ohm-_____.

(iii) For a given pair of coils, the magnetic flux linked with one of them, due to a unit current in the other coil, has the _____________ ____________ irrespective of which of the two coils carries the current.

(iv) The magnetic field can be regarded as a seat of _______, very much like the ________field.

(v) The self induced emf in a coil, always _____ the __________that is trying to set up a current in the coil.

Answers:

(i) current

(ii) /ampere; -second

(iii)same value

(iv)energy; electric

(v)opposes; source

True or False

State whether the following statements are true or false:

(i)The henry (H) is the common SI unit, both for the coefficient of self inductance as well as the coefficient of mutual inductance.

(ii) The coefficient of self inductance would have a value of 0.1 H when the magnetic flux linked with the coil due to a current of 10 A in it, equal 0.5 Wb.

(iii) The self inductance of a coil depends on the current flowing through it.

(iv)The coefficient of mutual inductance for a pair of coils depends on their distance of separation.

(v) The formula for M, that leads to a proof of the reciprocity theorem is known as the Newmann's formula.

Answers:

(i) True

(ii) False. We have $\phi = L I$. Hence $L = \phi/I = 0.5/10 = 0.05$

(iii) False. The self inductance is a characteristic of the coil itself and does not depend on the current flowing through it.

(iv) True

(v) True

Multiple Choice Questions

In the following questions each statement is followed by four choices only one of which is correct. You have to select that correct choice in each question.

(i) The SI unit of inductance, the henry, equals the self inductance of a coil for which the,

(a) magnetic flux linked with the coil due to a current of ½ A in it, equals 1Wb

(b) e.m.f, induced in it when the current in it changes at the rate of 2 A/s, equal 1 volt

(c) work done, by a source, in establishing a current of √2 A in it, equal 1 joule

(d) work done by a source in establishing a current of 2A in it, equal 1 joule.

Answer: (c)

Justification for the correct answer:

For choice (c), we have $W = \frac{1}{2} L I^2 = \frac{1}{2} \times 1 \times (\sqrt{2})^2 = 1$ Joule. Hence choice (c) is correct choice.

Choice (a) is incorrect as the correct value of current is 1 A.

Choice (b) is incorrect as the correct value of rate of change of current is 1 A/s.

Choice (d) is incorrect as the work done for L = 1H, would be $\frac{1}{2} \times 1 \times 2^2$ J = 2 J.

(ii) A source of voltage V=2V, sends a current of 0.5A for a time of t seconds. The energy spent by this source in time t equal the work needed to establish a current of 0.5A in a coil of self inductance 800mH. The time t equals:

(a)10^{-1} s

(b)10^{0} s

(c)10^{1} s

(d)10^{2} s

Answer: (a)

Justification for the correct answer:

We are given that

$V I t = \frac{1}{2} L I^2$

Therefore, $t = \frac{1}{2} L I / V$

$= \frac{1}{2} \times 800 \times 10^{-3} \times 0.5/2$ s

$= 10^{-1}$ s

Hence choice (a) is the correct choice.

(iii) Two coils of self inductance 400mH and 625 mH are kept in vicinity of each other. The coefficient of mutual inductance of these coils, say M, would then have a value such that:

(a) $0 \ll M \ll 500$ mH

(b) $0 \ll M \ll 512.5$ mH

(c) $0 \ll M \ll 625$ mH

(d) $0 \ll M \ll 1025$ mH

Answer: (a)

Justification for the correct answer:

We have $M = \sqrt{L_1 L_2}$ when the coefficient of coupling of the two coils is 1. Hence the maximum value of M is:

$(400 \times 625 \times 10^{-6})^{1/2}$ H = 500×10^{-3} H = 500 mH

Thus M lies between 0 and 500 mH. Hence choice (a) is the correct choice.

(iv) From the following, the only factor, on which the coefficient of mutual inductance, of a given pair of coils, does not depend, is:

(a) the geometry of the two coils

(b) the medium associated with the two coils

(c) the distance of separation of the two coils

(d) the current flowing in one of the two coils

Answer: (d)

Justification for the answer:

The coefficient of mutual inductance of a given pair of coils, depends on the geometry of the two coils, the medium associated with the two coils and the distance of separation of the two coils. It is a characteristic for a given pair of coils and does not depend on the current flowing in one of the two coils. Hence choice (d) is the correct choice.

(v) The work done in establishing a current I, in a coil of self inductance L, is stored in the magnetic field established in a volume τ, around the coil. The average value of the magnetic field, say B, in this volume τ, is:

(a) $B = \sqrt{\frac{\tau}{L\mu_0}}\, I$

(b) $B = \sqrt{\frac{L\mu_0}{\tau}}\, I$

(c) B = $\sqrt{L\tau\mu_0}$ I

(d)B = I/ $\sqrt{L\tau\mu_0}$

Answer: (b)

Justification for the correct answer:

The energy density of the magnetic field is $B^2/2\mu_0$. Hence the total energy stored in the volume τ, in which the magnetic field is present is $(B^2/2\mu_0)\ \tau$. Here B stands for the average value of the magnetic field. We thus have

$$(B^2/2\mu_0)\ \tau = \tfrac{1}{2} L I^2$$

Therefore, B = $\sqrt{\frac{L\mu_0}{\tau}}$ I

Hence choice (b) is correct choice.

Essay Type Questions

1. Define the unit henry in two different ways both for the case of (i) coefficient of self inductance and (ii) coefficient of mutual inductance.

2. Define and discuss the term coefficient of coupling. How does this term lead to an upper limit for M, in terms of L_1 and L_2 ?

3. Obtain an expression for the work to be done to establish a current I in a coil of self inductance L.

4. State and prove the reciprocity theorem.

5. Derive and expression for the coefficient of

(i) self inductance for a solenoidal coil

(ii) mutual inductance for a pair of closely wound solenoidal coils.