[This question paper contains 4 printed pages]

Your Roll No.............

H

Sr. No. of Question Paper : 3005

Unique Paper Code : 32351402

Name of the Paper : Riemann Integration and Series of Functions

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. Use of calculator is not allowed.

1. (a) Find the upper and lower Darboux integral for $f(x) = 2x + 1$ and show that it is integrable on $[1,2]$. Hence show that $\int_1^2 (2x + 1)dx = 4$. 6.5)

(b) Prove that a bounded function f on [a, b] is Riemann integrable if and only if it is Darboux

P.T.O.

integrable in which case the values of the integrals agree. (6.5)

(c) Prove that every continuous function f on [a, b] is integrable. (6.5)

2. (a) (i) State and prove intermediate value theorem for integrals.

(ii) Give example of a function f which is not integrable for which |f| is integrable. (6)

(b) If u and v are continuous functions on [a, b] that are differentiable on [a, b] and if u′ and v′ are integrable on [a, b] then prove

$$\int_a^b u(x)v'(x)+\int_a^b u'(x)v(x)=u(b)v(b)-u(a)v(a)$$

(6)

(c) Prove that if f is a piecewise continuous function or a bounded piecewise monotonic function on [a, b] then f is integrable on [a, b]. (6)

3. (a) Prove that $\int_0^\infty e^{-t}t^{s-1}dt$ is convergent if and only if $s > 0$. (6)

(b) (i) Prove that $\int_1^\infty \frac{\sin}{x^2}dx$ converges absolutely

(ii) Prove that $\int_1^{\infty} \frac{\sqrt{x}}{x^3+5} dx$ is convergent. (6)

(c) Test convergence of

(i) $\int_2^{\infty} \frac{2x^2}{x^4-1} dx$ (ii) $\int_0^{\infty} e^{-x^2} x^2 dx$ (6)

4. (a) Let $\langle f_n \rangle$ be a sequence of bounded functions on $A \subseteq R$. Then this sequence converges uniformly on A to a bounded function f if and only if for each $\in > 0$ there is a number $H(\in)$ in N such that for all $m, n \geq H(\in)$ then $||f_m - f_n||_A \leq \in$. (6.5)

(b) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \geq 0$, $n \in N$. Show that the sequence $\langle f_n \rangle$ converges non-uniformly on $[0, \infty)$ and converges uniformly on $[a, \infty)$, $a > 0$. (6.5)

(c) If a > 0, show that $\lim_{n \to \infty} \left(\int_a^{\pi} \frac{\text{Sin } nx}{nx} dx \right) = 0$. What happens if $a = 0$. (6.5)

5. (a) State and prove Weierstrass M-test for the uniform Convergence of a series of functions. (6.5)

(b) Discuss the convergence and uniform convergence of the series of functions

$$\Sigma(x^n + 1)^{-1}, \ x \neq 0$$

(6.5)

(c) If f_n is continuous on $D \subseteq R$ to R for each $n \in N$ and Σf_n converges to f uniformly on D then show that f is continuous on D. (6.5)

6. (a) (i) Find the radius of convergence of the power series (3)

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

(ii) Define sinx as a power series and find its radius of convergence. (3)

(b) If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$ also have radius of convergence R. (6)

(c) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ for } |x| < R.$$ (6)

[This question paper contains 4 printed pages.]

Your Roll No. ....

Sr. No. of Question Paper : 3116

Unique Paper Code : 32351403

Name of the Paper : Ring Theory & Linear Algebra-I

Name of the Course : **B.Sc. (Hons.) Mathematics CBCS (LOCF)**

Semester : IV

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts from each question.

1. (a) Prove that every finite Integral domain is a field. Give an example of an infinite integral domain which is not a field, Justify. (6)

   (b) (i) Let F be a field of order 2^n. Prove that the characteristic of F is 2.

   (ii) Find all the units in $\mathbb{Z}[i]$ (6)

   (c) Prove that the set of all the nilpotent elements of a commutative ring form a subring. (6)

P.T.O.

2. (a) Let R be a commutative ring with unity and A be an ideal of R then prove that R/A is an integral domain if and only if A is a prime ideal of R. (6)

(b) Prove that $\mathbb{Z}[i]/\langle 1-i\rangle$ is a field. (6)

(c) Find all the maximal ideals of $\mathbb{Z}_{20}$. (6)

3. (a) Find all the ring homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_{10}$. (6.5)

(b) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$ and Φ be the mapping that takes $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ to a. Show that

(i) Φ is a ring homomorphism

(ii) Determine the kernel of Φ.

(iii) Is Φ a one-one mapping. Justify. (6.5)

(c) State and prove first isomorphism theorem for rings. (6.5)

4. (a) Let V(F) be a vector space.

(i) Prove that the intersection of two subspaces of V(F) is also a subspace of V(F).

(ii) Show that union of two subspaces of V(F) may not be a subspace of V(F). Discuss the condition under which union of two subspaces will also form a subspace of V(F). (6)

(b) Let S be a linearly independent subset of a vector space V(F), and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent iff $v \in \text{Span}(S)$. (6)

(c) Let u, v, w be distinct vectors of a vector space V. Show that if {u,v,w} is a basis for V, then {u+v+w, v+w, w} is also a basis for V. (6)

5. (a) Let V(F) and W(F) be vector spaces and let $T: V \to W$ be a linear transformation. If V is a finite-dimensional, then

$$\text{Dim}(V) = \text{Nullity}(T) + \text{Rank}(T).$$ (6.5)

(b) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ and $U: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$$

$$U(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2)$$

Let β, γ be the standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, Prove that

(i) $[T+U]_\beta^\gamma + [T]_\beta^\gamma + [U]_\beta^\gamma$

(ii) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars a. (6.5)

(c) Let V and W be vector spaces. Let $T: V \to W$ be linear and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of Range of T. Prove that if $S = \{v_1, v_2, \ldots v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, \ldots, k$. Then S is linearly independent.

(6.5)

6. (a) Let T be a linear operator on a finite dimensional vector space V. Let β and β′ be the ordered basis for V. Suppose that Q is the change of coordinate matrix that changes β′ coordinates into β coordinates, then $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. (6.5)

(b) Let β, γ be the standard ordered basis of $P_1(\mathbb{R})$ and $\mathbb{R}^2$ respectively.

Let $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ be a linear transformation defined by $T(a + bx) = (a, a+b)$.

Find $[T]_\beta^\gamma$, $\left[T^{-1}\right]_\gamma^\beta$ and verify that $\left[T^{-1}\right]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$.

(6.5)

(c) Let $U: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ and $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformations respectively defined by

$U(f(x)) = f'(x)$ and $T(f(x)) = \int_0^x f(t)\,dt$. Prove that $[UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha$ where α and β are standard ordered basis of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively.

(6.5)

[This question paper contains 4 printed pages.] 30/5/24

Your Roll No..............

Sr. No. of Question Paper : 4014

Unique Paper Code : 2352572401

Name of the Paper : Abstract Algebra

Name of the Course : **B.A./B.Sc. (Programme) with Mathematics as Non-Major/ Minor – DSC**

Semester : IV

Duration : 3 Hours   Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** questions by selecting **two** parts from each question.

3. Each part carries **7.5** marks.

4. Use of Calculator not allowed.

P.T.O.

1. (a) Let n be a fixed positive integer greater than 1. If a mod n = a' and b mod n = b', then prove that (a + b)mod n = (a' + b')mod n and (ab)mod n = (a'b')mod n.

(b) Define a Group. Show that the set {5,15, 25, 35} is a group under multiplication modulo 40. What is the identity element of this group? Is there any relationship between this group and U(8)?

(c) Show that G = GL(2, $\mathbb{R}$), group of 2×2 matrices with real entries and nonzero determinants is non abelian. Also, construct a Cayley table for U(12).

2. (a) Let G be a group and H a nonempty subset of G. Prove that if ab^{-1} is in H whenever a and b are in H, then His a subgroup of G. Also, find the order of 7 in U(15).

(b) Let G be a group and let a ∈ G. Prove that a and a^{-1} have the same order. Illustrate the above result in the group $\mathbb{Z}_{10}$.

(c) Let a be an element of order n in a group G and let k be a positive integer. Show that

$$\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle \text{ and } |a^k| = \frac{n}{\gcd(n,k)}.$$

3. (a) If $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$.

Show that $\alpha\beta \neq \beta\alpha$. Also compute the value of and find order of $\beta^{-1}\alpha\beta$.

(b) Construct a complete Cayley table for D_4, the group of symmetries of a square. Find the inverse of each of the element in D_4.

(c) Let H = {(1), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)}. Find all the left cosets of H in A_4, the alternating group of degree 4.

4. (a) State Lagrange's theorem for finite groups. Show that a group of prime order is cyclic.

(b) State the normal subgroup test. Prove that $SL(2, \mathbb{R})$, the group of 2×2 matrices with determinant 1 is a normal subgroup of $GL(2, \mathbb{R})$, the group of 2×2 matrices with non zero determinant.

(c) Let $f: G \to G'$ be a group homomorphism. Prove that if H be a cyclic subgroup of G then f(H) is a cyclic subgroup of G'. Prove or disprove that the map $g: (\mathbb{R}, +) \to (\mathbb{R}, +)$ given by $g(x) = x^2$ is a homomorphism.

5. (a) (i) Prove that the set $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$ is a subring of the ring of all 2×2 matrices over $\mathbb{Z}$.

(ii) Prove that the ring $\mathbb{Z}_p$ of integers modulo p where p is a prime, is an integral domain.

(b) Define characteristic of a ring. Let R be a ring with unity 1. Prove that if 1 has infinite order under addition then charR = 0. If 1 has order n under addition, then charR = n.

(c) Prove that intersection of any collection of subrings of a ring R is a subring of R. Does the result hold for the union of subrings? Justify.

6. (a) State the ideal test. Let $\mathbb{R}[x]$ denote the set of all polynomials with real coefficients. Let $A = \{p(x) \in \mathbb{R}[x] \mid p(0) = 0\}$. Prove that A is a principal ideal of $\mathbb{R}[x]$.

(b) Determine all the ring homomorphisms from $\mathbb{Z}_n$ to itself.

(c) State and prove the First Isomorphism Theorem for rings.

[This question paper contains 8 printed pages.]

Your Roll No. 22019560021

28/5/24

Sr. No. of Question Paper : 4064 **H**

Unique Paper Code : 2352012401

Name of the Paper : Sequence and Series of Functions

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : IV

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

3. **All** questions are compulsory and carry equal marks.

1. (a) Define uniform convergence of a sequence of functions (f_n) defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$. If $A \subseteq \mathbb{R}$ and $\varphi: A \to \mathbb{R}$ then define the uniform norm of

φ on A. Discuss the uniform and pointwise convergence of the sequence (f_n), where

$$f_n(x) = \frac{x}{n} \text{ for } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

(b) Let (f_n) be the sequence of functions defined by

$$f_n(x) = \frac{1}{1+x^n} \quad \forall x \in [0,1],\ n \in \mathbb{N}.$$

Find the pointwise limit of the sequence (f_n). Does (f_n) converges uniformly? Justify your answer.

(c) Show that if (f_n) and (g_n) are two sequences of bounded functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ that converge uniformly to f and g respectively then prove that the product sequence $(f_n g_n)$ converge uniformly on A to fg. Give an example to show that in general the product of two uniformly convergent sequence may not be uniformly convergent.

2. (a) Let (f_n) be a sequence of integrable functions on $[a, b]$ and suppose that (f_n) converges uniformly to f on $[a, b]$. Show that f is integrable on $[a, b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

(b) Let $f_n(x) = \dfrac{x^n}{n}$ for $x \in [0, 1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on $[0, 1]$ and that the sequence (f_n') converges on $[0, 1]$ to a function g but the convergence is not uniform.

(c) Show that the sequence $\left(\frac{x^n}{1+x^n}\right)$ does not converge uniformly on [0,2].

3. (a) State and prove Weierstrass M-Test for uniform convergence of series of functions.

(b) Show that the series $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ is uniformly convergent on [–a, a], a > 0 but is not uniformly convergent on $\mathbb{R}$.

(c) Discuss the pointwise convergence of the series of functions $\sum_{n=1}^{\infty} \frac{x^n}{2+3x^n}$ for $x \geq 0$.

4. (a) Let f_n be continuous function on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in N$ and $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on D. Prove that f is continuous on D.

(b) Show that the series of functions $\sum_{n=1}^{\infty} \frac{\cos(x^2+1)}{n^3}$ converges uniformly on $\mathbb{R}$ to a continuous function.

(c) Given the Riemann integrable functions $f_n : \mathbb{R} \to \mathbb{R}$, $n \in N$, such that

$$f_n(x) = \sin\left(\frac{x}{n^4}\right) \text{ for all } x \in \mathbb{R}$$

Show that $\sum_{n=1}^{\infty} \int_{-2\pi}^{2\pi} f_n(x)\, dx = \int_{-2\pi}^{2\pi} \sum_{n=1}^{\infty} f_n(x)\, dx$.

5. (a) Define the radius of convergence and interval of convergence of power series. Check the uniform convergence of the following power series on [−1, 1]

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots\ldots\ldots\ldots$$

(b) State and Prove Cauchy-Hadamard Theorem.

(c) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then prove that the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ has also radius of convergence R and for $|x| < R$

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

6. (a) For cosine function C(x) and sine function S(x) prove the following :

(i) If $f: \mathbb{R} \to \mathbb{R}$ is such that $f''(x) = -f(x)$ for $x \in \mathbb{R}$ then there exist real numbers α and β such that $f(x) = \alpha C(x) + \beta S(x)$ for $x \in \mathbb{R}$.

(ii) If $x \in \mathbb{R}$, $x \geq 0$ then

$$1-\frac{1}{2}x^2 \leq C(x) \leq 1-\frac{1}{2}x^2+\frac{1}{24}x^4.$$

(b) State Abel's Theorem. Show that

$$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}\text{............} \quad -1 \leq x \leq 1$$

and $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\text{............}$.

(c) Prove that for every continuous function f on [0, 1], the sequence of polynomials $B_n f \to f$ uniformly on [0, 1], where $(B_n f)$ is the sequence of Bernstein's polynomials for the function f.

[This question paper contains 4 printed pages.] 30/5/24

Your Roll No.............

Sr. No. of Question Paper : **4102**

Unique Paper Code : 2352012402

Name of the Paper : Multivariate Calculus

Name of the Course : **B.A./B.Sc. (H)**

Semester : IV (DSC)

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator not allowed.

1. (a) Let f be the function defined by

$$f(x,y)=\begin{cases} \dfrac{xy^3}{x^2+y^6}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Is f continuous at (0,0)? Explain.

P.T.O.

(b) A cylindrical tank is 4 ft high and has an outer diameter of 2 ft. The walls of tank are 0.2 inches thick. Approximate the volume of interior of tank assuming the tank has a top and a bottom that are both 0.2 inches thick.

(c) Find an equation for each horizontal tangent plane to the surface

$$z = 5 - x^2 - y^2 + 4y.$$

2. (a) Let $f(x, y, z) = ye^{x+z} + ze^{y-x}$. At the point (2,2,–2) find the unit vector pointing in the direction of most rapid increase of f. And what is the value of most rapid increase of f?

(b) Find the absolute extrema of the function $f(x, y) = 2 \sin x + 5 \cos y$ on the set S where S is the rectangular region with vertices (0,0), (2,0), (2,5) and (0,5).

(c) Find the point on the plane $2x + y + z = 1$ that is nearest to the origin.

3. (a) (i) Sketch the region of integration and compute the double integral

$$\int_0^{\pi/2} \int_0^{\sin x} e^x \cos x \, dy dx .$$

(ii) Evaluate $\iint_D (2y-x)\,dA$ where D is the region bounded by $y = x^2$, $y = 2x$.

(b) Evaluate the area bounded by $r = 2\cos\theta$.

(c) Evaluate the given integral by converting to polar coordinates

$$I = \int_0^2 \int_y^{\sqrt{8-y^2}} \frac{1}{\sqrt{1+x^2+y^2}} dydx .$$

4. (a) (i) Set up a triple integral to find volume of solid bounded above by paraboloid $z = 6 - x^2 - y^2$ and below by $z = 2x^2 + y^2$.

(ii) Set up a double integral to find volume of solid region bounded by ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 .$$

(b) Use cylindrical co-ordinates to compute integral

$\iiint_D z(x^2+y^2)^{-1/2} dxdydz$ where D is the solid bounded above by plane $z = 2$ and bounded below by surface $2z = x^2 + y^2$.

(c) Find volume of solid D where D is the intersection of solid sphere $x^2 + y^2 + z^2 \le 9$ and solid cylinder $x^2 + y^2 \le 1$.

5. (a) Evaluate $\oint_C \left(x^2 y dx - xy\, dy\right)$, where C is the path that begins at (0,0), goes to (1,1) along the parabola $y = x^2$, and then return to (0,0) along the line $y = x$.

(b) Prove or disprove that the line integrals are path independent.

(c) Evaluate the line integral

$$\oint_C [(2x - x^2 y)e^{-xy} + \tan^{-1} y]i + [\frac{x}{y^2+1} - x^3 e^{-xy}]j.dR,$$

where C, the curve with parametric equations $x = t^2 \cos \pi t$, $y = e^{-t} \sin \pi t$, $0 \leq t \leq 1$.

6. (a) Find the area enclosed by the semicircle $y = \sqrt{4 - x^2}$ using the line integral.

(b) Find the mass of the homogenous lamina of density $\delta = x$ that has the shape of the surface S given by $z = 4 - x - 2y$ with $z \geq 0$, $x \geq 0$ and $y \geq 0$.

(c) Let $F = 2xi - 3yj + 5zk$, and let S be the hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with the disk $x^2 + y^2 \leq 9$ in the xy–plane. Verify the divergence theorem.

[This question paper contains 8 printed pages.]

Your Roll No.............

Sr. No. of Question Paper : 4140

H

Unique Paper Code : 2352012403

Name of the Paper : NUMERICAL ANALYSIS

Name of the Course : **B.Sc. (H) Mathematics – DSC**

Semester : IV

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. All **six** questions are compulsory, attempt any **two** parts from each question.

3. **All** questions carry equal marks.

4. Use of non-programmable Scientific Calculator is allowed.

P.T.O.

1. (a) Determine a formula which relates the number of iterations n, required by the bisection method to converge to within an absolute error tolerance ε, starting from the initial interval (a, b).

(b) Perform up to four iterations using the false position to approximate a root of $f(x) = e^x + x^2 - x - 4$ in the interval $(-2, -1)$.

(c) Verify that $x = \sqrt{a}$ is a fixed point of the function $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$. Use the fixed point iteration scheme to determine the order of convergence and the asymptotic error constant of the sequence $p_n = g(p_{n-1})$ towards $x = \sqrt{a}$.

2. (a) Calculate the fourth approximation of the root of the function $f(x) = e^{-x} - x$ using the Newton's method with $p_0 = 0$.

(b) Use the Secant method to calculate the root of the function $f(x) = 1.05 - 1.04x + \ln x$ using $p_0 = 1.10$ and $p_1 = 1.15$ with an absolute tolerance of 10^{-6} as a stopping condition.

(c) Determine the order of convergence of the Newton's method to find a root p of a twice differentiable function f on the interval [a, b], provided $f'(p) \neq 0$.

3. (a) Solve the following system of equations by using the LU Decomposition method

$$x + 4y + 3z = -4,$$

$$2x + 7y + 9z = -10,$$

$$5x + 8y - 2z = 9.$$

(b) Starting with initial vector (x, y, z, t) = (0, 0, 0, 0), perform three iterations of Gauss-Seidel method to solve the following system of equations

$$4x + y + z + t = -5,$$

$$x + 8y + 2z + 3t = 23,$$

$$x + 2y - 5z = 9,$$

$$-x + 2z + 4t = 4.$$

(c) Compute T_{jac} and T_{gs} for the matrix $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$.

Will the Jacobi and Gauss Seidel method converge for any choice of initial vector $x^{(0)}$? Justify your answer.

4\. (a) Find the Lagrange interpolation polynomial for the data set (0,1), (1,3) and (3,55). Also estimate the value at x = 2.5.

(b) The population of a town during the last six censuses are given below. Estimate the increase in the population from 1846 to 1848 by using the Newton interpolation polynomial.

Year	1811	1821	1831	1841	1851	1861
Population (in thousands)	9	12	20	17	37	42

(c) Obtain the piecewise linear interpolating polynomial for the function f(x) defined by the given data :

x	0	1	2	3	4
$f(x)$	1	2	5	10	16

Hence estimate the value of f(1.5), f(2.5) and f(3.5).

5. (a) Prove that $f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$ and determine the approximate value of the derivative of $f(x) = 1 + x + x^3$ at $x_0 = 1$, taking h = 1, 0.1, 0.01 and 0.001.

(b) Verify that the second-order forward difference approximation for the first derivative provides the exact value of the derivative, regardless of h, for the functions f(x) = 1, f(x) = x and $f(x) = x^2$, but not for the function $f(x) = x^3$.

(c) Derive the formula for the Trapezoidal rule and hence evaluate the approximate value of the integral $\int_1^2 \frac{1}{x} dx$.

6. (a) Determine the values for the coefficients A_0, A_1, and A_2 so that the quadrature formula

$$I(f) = \int_{-1}^{1} f(x)\, dx = A_0 f\left(\frac{-1}{3}\right) + A_1 f\left(\frac{1}{3}\right) + A_2 f(1)$$

has degree of precision at least 2.

(b) Use the Euler Method, determine the approximate solution of the initial value problem $\frac{dx}{dt} = 1 + \frac{x}{t}$, $x(1) = 1$, $1 \le t \le 2$, taking the step size as $h = 0.5$.

(c) Use the Runge-Kutta Method fourth-order, determine the approximate solution of the initial value Problem $\frac{dx}{dt} = \frac{x}{t}$, $x(0) = 1$, $1 \leq t \leq 2$, taking the step size as $h = 0.5$.

[This question paper contains 12 printed pages.]

10/6/24

Your Roll No..............

H

Sr. No. of Question Paper : 4227

Unique Paper Code : 2353012005

Name of the Paper : Mathematical Modeling

Name of the Course : **B.Sc. (H) – DSE**

Semester : IV

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Each question contains three parts. Attempt all questions by selecting any two parts from each question.

3. Parts of questions to be attempted together.

4. **All** questions carry equal marks

5. Use of non-programmable scientific calculators is allowed.

P.T.O.

1. (a) (i) Define Mathematical modeling. Explain the modeling cycle process. A ball and bat together cost \$1.10. The bat costs \$1.00 more than the ball. Model this problem by using mathematical symbols and determine how much does the ball cost.

(ii) Compute the dimensions of a, D, λ_1 and λ_2 from the following differential equation, assuming that it is a dimensionally consistent equation.

$$\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} + \lambda_1\sqrt{u} - \lambda_2 u^2$$

(b) In a particular epidemic model, where the infected individuals eventually recover, the population dynamics are governed by the following system of differential equations :

$$\frac{dS}{dt} = -\frac{\beta SI}{N}, \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I.$$

Given the parameter values $\beta = 2$ and $\gamma = 0.4$ and assuming that the total population N is 1000, in which initially there is only one infected individual but there are 999 susceptible individuals within the population, answer the following :

(i) Using the given parameters, calculate the basic reproduction number (r_0).

(ii) What is the peak number of infected individuals at any given time during the epidemic?

(c) Consider the following susceptible-infected-recovered (SIR) model :

$$\frac{dS}{dt} = -\beta\sqrt{S}\sqrt{I}, \frac{dI}{dt} = \beta\sqrt{S}\sqrt{I} - \gamma\sqrt{I}, \frac{dR}{dt} = \gamma\sqrt{I},$$

where β and γ are constant parameters and initial subpopulations are given as $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, $R(0) = 0$. By taking the first two equations of the model for analysis purposes, and assuming the transformation $u = \sqrt{S}$, $v = \sqrt{I}$, show that

$$v(t) = \left[\left(\sqrt{S_0} - \sqrt{S^*}\right)^2 + I_0\right]^{\frac{1}{2}} . \sin\left(\frac{\beta}{2}t + \phi_1\right),$$

where $\phi_1 = \tan^{-1}\left(\frac{\sqrt{I_0}}{\sqrt{S_0}-\sqrt{S^*}}\right)$ and $S^* = \frac{\gamma}{\beta}$.

2. (a) Consider an epidemic model, where the infected individuals eventually recover and the dynamics of the population are described by ≥ differential equations :

$$\frac{dS}{dt} = -\beta\sqrt{S}\sqrt{I}, \frac{dI}{dt} = \beta\sqrt{S}\sqrt{I} - \gamma\sqrt{I}.$$

Using the parameter values $\beta = 0.02$ and $\gamma = 0.4$ and assuming initially there is only one infected individual but there are only 500 susceptible individuals within a population,

(i) Calculate the basic reproduction number (r_0) using the given parameters.

(ii) How many individuals remain susceptible and never get infected throughout the epidemic?

(iii) What is the maximum number of individuals infected at any point in time during the epidemic?

(b) Construct a Susceptible-Exposed-Infectious-Recovered (SEIR) model using a system of first-order differential equations to describe the spread of disease. Divide the total population N(t) at any time t into subpopulations according to disease status. Assuming a constant influx of individuals into the susceptible population and a natural death rate affecting each subpopulation, as well as the exposure rate of susceptible individuals and recovery rate of infectious people, obtain the exact solution for the total population N(t), and provide the discretized form of the model equations using the Nonstandard Finite Difference (NSFD) scheme.

(c) Consider the following susceptible-infected-recovered (SIR) model :

$$\frac{dS}{dt} = -\beta S\frac{I}{N}, \frac{dI}{dt} = \beta S\frac{I}{N} - \gamma I, \frac{dR}{dt} = \gamma I,$$

where β and γ are constant parameters and initial subpopulations are :

$S(0) = S_0 > 0,\ I(0) = I_0 > 0,\ R(0) = R_0 \geq 0.$

(i) Find the condition for epidemic occurrence by finding the basic reproduction number (r_0).

(ii) If the first integral expression in the (S, I) plane is given as $S(t) + I(t) = I_0 + S_0 + S^* \ln\left(\frac{S(t)}{S_0}\right)$, where $S^* = \frac{\gamma}{\beta}N$, then find the expressions for the maximum infective number (I_{max}), and the number of susceptibles who do not succumb to the epidemic (S_∞).

3. (a) Explain and analyze the Predator-Prey system (Lotka-Volterra model) by identifying and discussing all isolated critical points.

(b) Find all critical points of the system of differential equations

$$\begin{cases} \frac{dx}{dt} = 1 - xy \\ \frac{dy}{dt} = x - y^3 \end{cases}$$

and determine (if possible) whether they are stable or unstable.

(c) Consider a damped nonlinear spring-mass system. Let m denote the mass of an object attached to a spring and let x(t) denote the displacement of the mass at time t from its equilibrium position. Assume that the mass on the spring is connected to a dashpot, exerting a force of resistance proportional to velocity, $y = x' = \frac{dx}{dt}$ of the mass. Let the force exerted by the spring on the mass be given by by $F(x) = m\frac{dy}{dt} = mx''$. The equation of motion of the mass is given by $mx'' = -cx' - kx +$

βx^3, where c, m, k, $\beta > 0$. If $c = 2$, $\beta = \frac{5}{4}$, $k = 5$, and $m = 1$, then write the corresponding system of first-order differential equations and discuss the stability of all critical points of the system.

4. (a) Consider the system of differential equations :

$$\begin{cases} \frac{dx}{dt} = x + \epsilon y \\ \frac{dy}{dt} = x - y \end{cases}.$$

Determine the conditions on e for which the critical point of the system may be a saddle point and a center in the phase plane and identify the points where this change occurs.

(b) Determine the type and stability of the critical point (x*, y*) for the following systems :

(i) $\frac{dx}{dt} = 33 - 10x - 3y + x^2, \frac{dy}{dt} = -18 + 6x + 2y - xy;$

Critical point: (x*,y*) = (4,3).

(ii) $\frac{dx}{dt} = 3x - x^2 - xy, \frac{dy}{dt} = y + y^2 - 3xy;$

Critical point: $(x^*, y^*) = (1,2)$.

(c) Consider the following system of differential equations :

$$\begin{cases} \frac{dx}{dt} = 60x - 3x^2 - 4xy \\ \frac{dy}{dt} = 42y - 3y^2 - 2xy \end{cases}$$

Show that the linearization of the system at (20,0) is

$$\begin{cases} u' = -60u - 80v \\ v' = 2v \end{cases} \text{ where } u' = \frac{du}{dt} \text{ and } v' = \frac{dv}{dt}.$$

Find the eigenvalues and corresponding eigenvectors of the coefficient matrix of the linear system. Hence, confirm the nature of the critical point (20,0).

5. (a) Using Monte Carlo simulation write an algorithm to calculate an approximate area trapped between the curves $y = x$ and $y^2 = 2x$.

(b) Explain the Linear Congruence Method of generating random numbers. List the drawbacks of this method. Find 10 random numbers using a = 5, b = 1, c = 8 and 7 as the seed number.

(c) Using Monte Carlo simulation, write an algorithm to calculate that part of the volume of an ellipsoid $\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{8} \leq 16$ that lies in the first octant, $x > 0$, $y > 0$, $z > 0$.

6. (a) Use the Simplex Method to solve the following problem

Max. $z = 3x + 2y$

s. to $-x + 2y \leq 4,$

$5x + 2y \leq 14,$

$x - y \leq 3,$

$x, y \geq 0.$

(b) Consider the following Linear Programming problem :

Max. $z = x + y$

s. to $x + y \leq 6,$

$3x - y \leq 9,$

$x, y \geq 0.$

Using algebraic methods find the possible number of points of intersection in the xy- plane. Are all the points feasible? If not, then how many are feasible and how many are infeasible. List the feasible extreme points along with the value of the objective function

(c) Solve the following Linear Programming problem graphically.

Max. $z = 20x + 30y$

s. to $3x + 2y \leq 80,$

$2x + 4y \leq 120,$

$x, y \geq 0.$

Perform a sensitivity analysis to find the range of values for the coefficient of x in the objective function for which the current extreme point remains optimal.